CARTIER DIVISORS

Abstract. This document is meant to be a clarification on the notion of Cartier divisors.

1. Cartier Divisors

1.1. **Meromorphic functions as sections of line bundles.** The following generalize the idea of meromorphic functions from complex geometry. Let $(X, \mathcal{O}_X)$ be a ringed space. We define for an open U

$$\mathcal{S}(U) = \{s \in \mathcal{O}_X(U) \mid \forall x \in U \quad s_x \in \mathcal{O}_{X,x} \text{ not a divisor of zero}\}$$

Note that we have obvious maps $\mathcal{S}(U) \to \mathcal{S}(V)$ if $U \subset V$, and that each $\mathcal{S}(U)$ is a multiplicative subset of $\mathcal{O}_X(U)$.

Definition 1.1 (Meromorphic functions)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. We define $\mathcal{K}_X$ the sheaf of meromorphic functions as the sheaf associated to the following presheaf :

$$U \to \mathcal{S}(U)^{-1}\mathcal{O}_X(U)$$

Note that we have a natural injection : $\mathcal{O}_X \to \mathcal{K}_X$.
The following definitions and propositions holds in a vast setting.

Definition 1.2 (Cartier Divisors)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. Consider the short exact sequence of abelian sheaves :

$$1 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 1$$

The group of Cartier divisors $\mathrm{CaDiv}(X)$ is defined to be : $\mathrm{H}^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. A global section can be represented as a collection of pairs (f_i, U_i) where $X = \bigcup U_i$ and $f_i \in \mathcal{K}(U_i)^\times$, such that $\frac{f_i}{f_j} \in \mathcal{O}(U_{ij})^\times$. The Cartier class group $\mathrm{CaCl}(X)$ is defined as the cokernel of $\mathrm{H}^0(X, \mathcal{K}_X^\times) \to \mathrm{H}^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$

We can think as a Cartier divisor (f_i, U_i) as the “zero set” of the zeroes and poles of the f_i’s counted with multiplicities.
The short exact sequence of abelian sheaves

$$1 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 1$$

gives a connecting morphism (which factors through $\mathrm{CaCl}(X)$)

$$\mathcal{O}(-) : \mathrm{CaDiv}(X) \to \mathrm{Pic}(X)$$

Using Čech cohomology we see that this morphism sends a divisor $D = (f_i, U_i)$ to the cocycles $(U_{ij}, \frac{f_j}{f_i})$ (or it’s inverse, as we are dealing with abelian sheaves, both maps fit in the exact sequence so this does not matter for our purposes). This can be interpreted as the line bundle $\mathcal{O}(D)$ defined on U_i by $\frac{1}{f_i}\mathcal{O}_{U_i}$. We will make more precise this last statement in the following lemmas.

Lemma 1.1. *Let D be a Cartier Divisor. Let $U = \bigcup U_i = \bigcup V_\alpha$ be two coverings such that $D = (f_i, U_i) = (g_\alpha, V_\alpha)$. Then*

$$\{s \in \mathcal{K}(U) \mid \forall i \quad sf_i \in \mathcal{O}_X(U_i)\} = \{t \in \mathcal{K}(U) \mid \forall \alpha \quad tg_\alpha \in \mathcal{O}_X(V_\alpha)\} \subset \mathcal{K}(U)$$

Proof. The equality $(f_i, U_i) = (g_\alpha, V_\alpha)$ means that there exists common refinement $(W_{\alpha ij})$ of both covers such that for all i, α, j we have :

$$\frac{f_i}{g_\alpha} \in \mathcal{O}_X(W_{\alpha ij})^\times$$

So we get

$$\begin{aligned} sf_i &\in \mathcal{O}_X(U_i) \\ \Rightarrow \forall j \quad sf_i &\in \mathcal{O}_X(W_{\alpha ij}) \\ \Rightarrow \forall j \quad sg_\alpha &\in \mathcal{O}_X(W_{\alpha ij}) \\ \Rightarrow sg_\alpha &\in \mathcal{O}_X(V_\alpha) \end{aligned}$$

has the setting is symmetric we get our claim. $\square$

Proposition 1.2 (Associated line bundle to a Cartier divisor)**.** *The morphism*

$$\mathcal{O}(-) : \mathrm{CaDiv}(X) \to \mathrm{Pic}(X)$$

can be realized by seeing sections of these line bundles as meromorphic functions

$$\mathcal{O}(D)(U) = \{s \in \mathcal{K}(U) \mid \forall i \quad sf_i \in \mathcal{O}_X(U_i)\} \subset \mathcal{K}(U)$$

for any $U = \bigcup U_i$ such that $D = (f_i, U_i)$.

Proof. If D is given by $f_i \in \mathcal{K}(U_i)^\times$, then we have $\mathcal{O}(D)(U_i) = \frac{1}{f_i}\mathcal{O}_X(U_i)$ and the section $\frac{1}{f_i} \in \mathcal{O}(D)(U_i)$ gives a trivialization. Moreover we have :

$$\begin{tikzcd} \mathcal{O}(D)_{U_{ij}} & \\ \mathcal{O}_{U_{ij}} \arrow[u, "f_i^{-1}"] \arrow[r, "\frac{f_j}{f_i}"'] & \mathcal{O}_{U_{ij}} \arrow[ul, "f_j^{-1}"'] \end{tikzcd}$$

so the cocycles of the line bundle $\mathcal{O}(D)$ are $(\frac{f_j}{f_i})$ as wanted. $\square$

Proposition 1.3. *If X is an integral scheme, $\mathcal{O}(-)$ defines an isomorphism*

$$\mathrm{CaCl}(X) \to \mathrm{Pic}(X).$$

Proof. If X is integral $\mathcal{K}_X^\times$ is the constant sheaf with value $K(X)$, therefore flabby, therefore acyclic. The claim now follows from the definition of CaCl and the long exact sequence in cohomology. $\square$

Remark. The way to interpret sections of $\mathcal{O}(D)$ is that they are meromorphic functions that *can have poles on D*. This means that if D is given locally by $\frac{f}{g}$ with f and g are functions in $\mathcal{O}$, then functions in $\mathcal{O}(D)(U)$ can have poles where f have zeroes and has to have zeroes where g has zeroes.

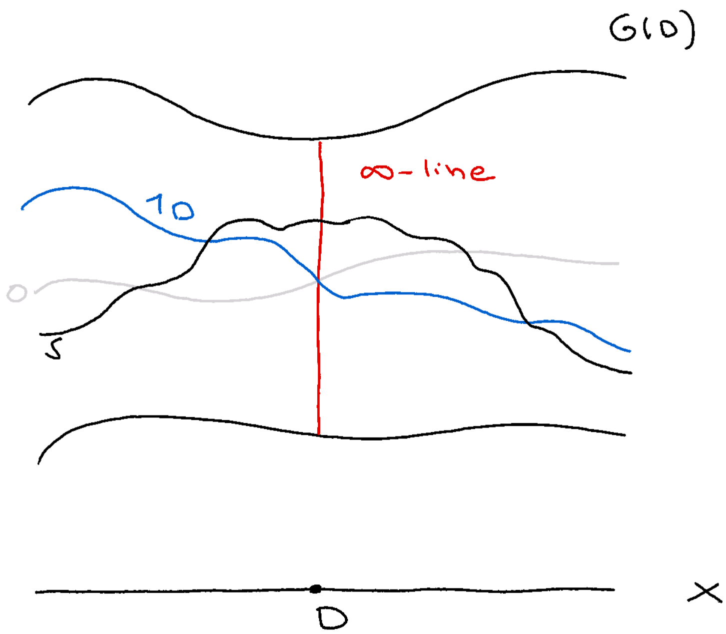

FIGURE 1. The section s can be interpreted as a meromorphic function with a pole on D.

Remark. We can understand how "sections of line bundles" can be seen as meromorphic functions with the following picture. Imagine that locally a Cartier D is given by $f \in \mathcal{O}(U)$. Then note that $1 \in \mathcal{K}(U)$ is in $\mathcal{O}(D)(U)$. In the following picture we drew this section as 1_D. We will see in the next section that 1_D vanishes exactly on $V(f) = D$. Now, sections of this line bundle on U can be viewed as meromorphic functions by the following: where 1_D does not vanish (does not intersect the zero section), it gives a trivialization of the fiber, so outside of the fiber of D, we can associate the section to a function to $\mathbb{A}^1$. More precisely let s be a section of $\mathcal{O}(D)$. Then at every point $x \in X \setminus D$ we have that

$$1_D : k(x) \xrightarrow{\sim} \mathcal{L}(x)$$

if we denote the inverse map by $\frac{-}{1_D}$ then $\frac{s(x)}{1_D}$ is a legitimate number in $k(x)$. We even have an isomorphism on $U = X \setminus D$

$$1_D : \mathcal{O}_U \xrightarrow{\sim} \mathcal{L}_U$$

so $\frac{s_U}{1_D} \in \mathcal{O}_U$ is a morphism to $\mathbb{A}^1$.

We called in the drawing where 1_D and the zero section cross (the fiber over D) the ∞-line: if a section cross this line not intersecting the zero section then the "associated function" will have a pole on D. If the section cross the zero section in the fiber of D then we do not know what happens : the value of the function can be 1, 0, any scalar or again a pole depending on how the section approaches the intersection.

1.2. **Effective Cartier Divisors and line bundles with a non-zero divisor section.** We now concentrate our attention to Cartier divisors that have "only positive multiplicities" so are "truly" geometric.

Definition 1.3 (Effective Cartier Divisor)**.** A Cartier divisor $D = (f_i, U_i)$ is said to be *effective* if $f_i \in \mathcal{O}(U_i)$ for every i.

Remark. If $D = (f_i, U_i)$ is effective, then by looking at the ideal sheaf generated by f_i's in $\mathcal{O}_X$ we see that Effective Cartier Divisors are in one to one correspondence with ideal sheaves that are line bundles. In what follows $(X, \mathcal{O}_X)$ is a scheme.

The above remark shows that effective Cartier Divisors are in one to one with the following concept.

Definition 1.4 (Geometric Cartier Divisor)**.** Let X be a scheme. A Geometric Cartier divisor is a closed subscheme $V(\mathcal{I})$ such that $\mathcal{I}$ is line bundle.

Definition 1.5 (Closed subscheme associated to a section of a line bundle)**.** Let $\mathcal{L}$ be a line bundle on a scheme X and let $s \in \mathrm{H}^0(X, \mathcal{L})$ be a global section. We define $V(s)$ as the closed subcheme associated to the following ideal sheaf :

$$\mathcal{I}_s = \mathrm{im}(\mathcal{L}^\vee \xrightarrow{\mathrm{ev}_s} \mathcal{O}_X)$$

Remark. The following lemma shows that this is indeed the tempting definition - note that the above definition gives a scheme and not only a closed topological space.

Lemma 1.4. *Topologically we have* $V(s) = \{x \in X \mid s(x) = 0\}$

Proof. The assertion can be checked locally on X so we can suppose that X is affine say equal to $\mathrm{Spec}(A)$ and that $\mathcal{L}$ is trivial. So applying the definition, an element $a \in A$ is sent to :

$$A^\vee = \mathrm{Hom}(A, A) \to A$$

$$(1 \mapsto x) \mapsto xa$$

so we get the claim because the image is precisely (a). □

Definition 1.6 (Non-zero divisor section)**.** Let $\mathcal{L}$ be a line bundle on X. A section $s \in \mathcal{L}(X)$ is said to be a non-zero divisor if :

$$\mathcal{O}_X \xrightarrow{s} \mathcal{L}$$

is injective.

So assembling definition we get the following lemma :

Lemma 1.5. *A section $s \in \mathcal{L}(X)$ is a non-zero divisor if and only if $\mathcal{L}^\vee \xrightarrow{\mathrm{ev}_s} \mathcal{O}_X$ is injective, and the associated closed subscheme is a geometric Cartier divisor.*

Remark. Note that for any geometric Cartier divisor or equivalently for any effective Cartier divisor, the line bundle $\mathcal{O}(D)$ described in the preceding section has a canonical global section 1_D, where we take this notation for the meromorphic function 1 in $\mathcal{O}(D)(X)$.

Lemma 1.6 (Any Geometric Cartier has can be realized has the vanishing locus of a global section of a line bundle)**.** *Let D be an effective Cartier divisor. Then the natural section 1_D described above is a non-zero divisor and we have $V(1_D) = D$ if we see D as a geometric Cartier divisor.*

Proof. If locally D is given by a function f_i, then note that trivializing and dualizing, evaluating at 1_D is evaluating at f_i, thus finishing the proof. □

The next proposition is the main result of this section :

Proposition 1.7 (Correspondence between Geometric Cartier Divisors and line bundles with a fixed non-zero divisor section)**.** *Let X be a scheme. Then we have the following correspondence :*

$$\{\textit{Effective Cartier divisors on } X\} \leftrightarrow \{(\mathcal{L}, s) \mid \mathcal{O}_X \xrightarrow{s} \mathcal{L}\}/\mathcal{O}_X^\times(X)$$
$$D \mapsto (\mathcal{O}(D), 1_D)$$
$$(\frac{s}{\varphi_i}, U_i) \leftarrow\!\shortmid (\mathcal{L}, s)$$

where $\frac{-}{\varphi_i}$ means the image by the inverse of a local trivialization $\mathcal{O}_X \xrightarrow{\varphi_i} \mathcal{L}$.

Proof. A precision: pairs $(\mathcal{L}, s)$ are up to isomorphism of the source $\mathcal{O}_X$ and the target $\mathcal{L}$. That is what we meant be quotienting out by $\mathcal{O}_X^\times(X)$.
Going from left to right to left is the identity because the image of 1_D by the trivialization are exactly the f_i's.
Note that going from right to left is well defined because a Cartier will not change by a global multiplication by an element of $\mathcal{O}_X(X)^\times$. As s is a a non-zero divisor section, $\frac{s}{\varphi_i}$ is an element of $\mathcal{S}(U_i)$ so invertible in $\mathcal{K}(U_i)$. Moreover, we have $\frac{s}{\varphi_i}/\frac{s}{\varphi_j}$ to be $\frac{\varphi_j}{\varphi_j}$ the cocycles of $\mathcal{L}$ so indeed in $\mathcal{O}_X(U_{ij})^\times$. Now, note that the desired isomorphism of line bundles folows because $\mathcal{L}$ and $\mathcal{O}(V(s))$ have the same cocycles, and the isomorphism constructed in this way sends 1_D to s. □

This result is geometrically soothing. Take any algebraic variety X and a codimension 1 closed subvariety D in it which happens to be a Cartier divisor. This result tells us that we can always thicken by lines in a clever way our variety X (the line bundle $\mathcal{O}(D)$ where X lives as the zero section) and always find another copy of X in this thickening by lines (the section 1_D) such that the intersection of the two copies are exactly D, in and this, in both copies.

Example (Incidence divisor)*.* Let X be a variety over a field k, such that $\mathrm{H}^0(X, \mathcal{O}_X) = k$, for example if k algebraically closed or if X is a complete intersection in a projective space. Or more generally: X being proper and geometrically connected. Let $\mathcal{L} \in \mathrm{Pic}(X)$ and $V \subset \mathrm{H}^0(X, \mathcal{L})$ be a finite dimensional linear subspace. Define $|V| = \mathrm{Proj}(S^*(V^*))$. This is the moduli space of geometric Cartier divisors of the form $V(s)$ for $s \in V$.
The goal is to define a divisor Γ in the product $X \times_k V$ such that the k-rational points of the divisor are :

$$\Gamma(k) = \{(x, V(s)) \in X(k) \times |V|(k) \mid x \in V(s)\}$$

that are pairs of a k-rational points in X together with a divisor $V(s)$ that contains it. We describe the construction in what follows.

Consider the box product $\mathcal{L} \boxtimes \mathcal{O}(1) = p_X^* \mathcal{L} \otimes p_V^* \mathcal{O}(1)$ and recall that by Künneth formula we have : $\mathrm{H}^0(X \times |V|, \mathcal{L} \boxtimes \mathcal{O}(1)) = \mathrm{H}^0(X, \mathcal{L}) \otimes_k \mathrm{H}^0(|V|, \mathcal{O}(1)) = \mathrm{H}^0(X, \mathcal{L}) \otimes_k V^*$. Let $\sigma \in V \otimes V^*$ be the element corresponding to the identity via the natural isomorphism $V \otimes V^* = \mathrm{Hom}(V, V)$. Then we define Γ by $V(\sigma)$.
We check that this is indeed what we wanted on k-rational points. Let $s_0, \dots, s_n$ be a basis of V. We have for $(x, V(s)) \in X(k) \times |V|(k)$, with $s = \sum_{i=1}^n \lambda_i s_i$, that :

$$0 = \sigma(x, V(s)) = \sum_{i=1}^n s_i(x) \otimes s_i^*(V(s))$$

that up to an isomorphism is identified with

$$0 = \sum_{i=1}^n \lambda_i s_i(x) = s(x).$$